# Two-sided Unobservable Investment, Bargaining, and Efficiency

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**Abstract**

We investigate bilateral bargaining when players can make preplay unobserved investments in the value of the item. With one-sided hidden investments, Gul (2001) shows that when bargaining inefficiencies are present underinvestment and strategic uncertainty occur. In our context of two-sided hidden actions, strategic uncertainty induces a post-investment bargaining problem with two-sided private information that mirrors Myerson and Satterthwaite (1983), and we would expect inefficiency. However, even though the bargaining protocol cannot be efficient in the presence of strategic uncertainty we find that unobserved investing and trade does not lead to distortions. The two potential sources of inefficiency offset each other. Equilibrium beliefs that in the presence of strategic uncertainty constrained optimal (second-best) trading will occur results in an unravelling effect absent in Gul. When both distortions are possible only equilibria in which neither the hold-up problem nor Myerson-Satterthwaites' logic emerge.

Keywords: bargaining, strategic uncertainty, hold-up
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# 1 Introduction

In an important synthesis of the hold-up problem and bargaining literature Gul (2001) shows that the magnitude of the inefficiency resulting from hold-up is related to the efficiency of the equilibrium of the bargaining game. If the seller makes a one-shot offer to a buyer who makes a relationship-specific investment before trade, then the inefficiency resulting with unobservable investment is equivalent to that resulting with observable investment (Gibbons, 1992). On the other end of the spectrum, when the seller makes repeated offers, and the time between offers vanishes, the investment decision of the buyer converges to the efficient level. Thus, Gul shows that if the equilibrium to the bargaining protocol extracts all the surplus, which is the case in the one-sided repeated offers game with one-sided incomplete information and vanishing time between periods (Gul and Sonnenschein, 1988), then the underinvestment associated with the hold-up problem goes away. His result also demonstrates that when bargaining is itself not fully-efficient (as in the case of non-trivial time-frictions) the presence of hidden investment decisions leads to additional distortions through the hold-up problem.

In this paper we investigate whether this intuition carries over to the case of two sided unobserved investments, a context in which the celebrated Myerson and Satterthwaite (1983) result suggests that bargaining cannot be efficient. Specifically, we investigate a setting in which a buyer and seller can make unobserved investments in the value of an indivisible item and then interact the kind of trading setup studied by Myerson and Satterthwaite. In this case strategic uncertainty on the part of both players generates a post-investment bargaining problem with two-sided private information. As is well-known, such bargaining environments often result in conflicts between efficient trade and incentive-compatibility in the trading mechanism. One might guess that, as in the case of the one shot game with one-sided incomplete information, the underinvestment problem persists because the bargaining protocol cannot be efficient if both the buyer and seller possess (endogenously generated) private information. Alternatively, one might guess that the inefficiencies from Myerson and Satterthwaite are persistent in settings where players induce strategic uncertainty through unobserved investments. We show that these conjectures fail. In particular, there are no equilibria in which investment decisions induce "second stage" post-investment bargaining inefficiencies. As a result, investment and trade are always optimal in the sense that after bargaining the good is always possessed by the player who values it more and the final owner has invested optimally given that she owns the good. This conclusion stems from an unraveling effect that undermines any putative mixed investment equilibrium that leads to the optimal second stage bargaining mechanism being inefficient. In other words only the natural equilibria exist–where either the buyer will invest optimally and obtain the item with probability one or the seller will invest optimally and retain the item with probability one.

Our approach is to connect with extant work as much as possible. We augment, in a way that is familiar in the hold-up literature, Myerson and Satterthwaite's canonical description of bilateral trade by allowing players' valuation of the traded item to result from their unobserved investments. When mixed strategies are played this causes asymmetric information to emerge endogenously. Specifically, when players cannot observe each other's investment decisions, mixing in the investment stage induces strategic uncertainty and, therefore, asymmetric information at the bargaining stage.[1] As is the case in other work of this form, equilibrium conjectures will lead players to believe that they know the distribution from which unobserved choices emerge. We then proceed to analyze what is possible in bargaining using the approach and many results from Myerson and Satterthwaite. We do need to provide technical extensions to their characterization to cover the case of poorly-behaved distributions but we relegate these details to the appendix.

In our framework there are three possible forms of inefficiency because both buyers and sellers can invest. First, at the *interim* bargaining stage, taking the investment decisions as fixed but unobserved, trade may exhibit the inefficiency that is central to Myerson Satterthwaite. Second, the eventual winner of the item may not have made an investment decision that would be optimal if she knew she were guaranteed to obtain the item. Third, even if the previous two forms of inefficiency are absent, the winning agent might not be the player that can obtain the greatest value from owning and optimally investing in the good.

[1] This is precisely the formulation used in Gul's case with one-sided hidden actions.

We find that the possibility of the first two forms of inefficiency offset each other and in fact, under the assumption that players anticipate the use of a rule that is "optimal" given beliefs resulting from equilibrium mixing probabilities, there are never equilibria with strategic uncertainty and inefficiencies from bargaining–so the first and second form of inefficiency do not obtain. This is true because given equilibrium beliefs about valuations and participation constraints, if players anticipate the use of a second-best trading rule, then either the strategic uncertainty that emerges will not lead to allocation inefficiencies or strategic uncertainty will not emerge. This is our main result, Theorem 4. The intuition behind this result can be obtained by considering the optimal Myerson Satterthwaite mechanism for the case where both the buyer's and seller's valuations are independent draws from the same uniform distribution. In the second-best rule certain types of buyers do not trade with any seller; this is precisely the source of the famous wedge. But if valuations are the result of strategic decisions we would not expect buyers to be willing to expend resources in order to obtain these particular valuations. It is better to not invest then to pay to obtain a valuation that does not trade. Thus, some of the valuations in the conjectured support cannot be optimal investment choices. The emergence of mixed valuations given by this distribution function is then not possible given an expectation that bargaining is described by a second-best mechanism.

Unraveling of this sort is not specific to the conjecture that equilibrium investment decisions induce uniform distributions over the valuations. It turns out to be pervasive regardless of the candidate equilibrium beliefs. Although it is possible to support lotteries over valuations which have overlapping supports, these distributions will not actually satisfy the conditions in Myerson Satterthwaite, and efficient allocations will be possible in the bargaining problem. Our conclusion is that knowledge that the bargaining mechanism is chosen optimally, given the relevant constraints and equilibrium beliefs about the investment strategies, implies that the form of allocation inefficiency that emerges in Myerson Satterthwaite is not consistent with equilibrium play. Moreover, the investment decisions will typically be in pure strategies and will be optimal, given the identity of the player getting the item on the equilibrium path. These two features of the equilibrium support a somewhat Coasian view where trade is generally efficient.

In addition to the efficiency result, along the way we provide two new technical results: we characterize the relationship between bargaining protocols and investment incentives and show that the Myerson and Satterthwaite Theorem fails to extend to the case of distributions of types with atoms, gaps, and connected components. Using an example we show that efficient and individually rational mechanisms exist when the distributions over valuations have gaps and atoms that have sufficiently large mass.

## 2 Model

Our point of departure from existing theory is to model the buyer and seller's valuations of the indivisible good as a function of investment decisions. For example, suppose that the object in question is a computing technology such as a search algorithm or mapping software and the potential owners are two competing technology companies. Each potential owner could make investments in the ability to interface the new technology with its existing products. Each could also invest time or money in finding alternatives to the technology in question. These investments then influence the value of the trade to each player. If there is no trade, the seller can capitalize on his investment but investment returns are lost to him if the object is sold. The opposite is true for the buyer; her investment generates value only when she purchases the good.

Formally, consider a risk-neutral seller (player $s$) who owns an indivisible object and a buyer (player $b$) who may wish to acquire it. Before trade/bargaining takes place the seller and the buyer can make unobserved relationship-specific investments $v_s$ and $v_b$. The value to player $i \in \{s, b\}$ of owning the item at the end of the buyer and seller's interactions is then $v_i - c_i(v_i)$. The cost function, $c_i(v_i)$, is strictly increasing (except possibly at the point 0), strictly convex and differentiable.[2] We assume that $c_i(0) = 0$ for both buyer and seller. We also assume that if either player knew she were

[2]We sometimes refer to these cost functions as the exogenous investment technologies.

going to own the item, her optimal investment would be finite, namely that for some finite level $\overline{\overline{v}}_i$ we have $c_i'(\overline{\overline{v}}_i) = 1$.

An investment strategy for a player is a choice of investment level. We allow the players to select mixed/behavioral strategies, so that the strategy for player $i$ is a cumulative distribution function $F_i(\cdot)$ over valuations (non-negative reals). These investments are assumed to be unobservable hidden actions.[3]

After the investment stage, the players interact and ultimately the item ends up owned by the buyer or seller and a transfer is made. We follow the approach in Myerson and Saterthwaite and abstract away from the particulars of the bargaining protocol and equilibrium descriptions. Instead we rely on a direct bargaining mechanism and incentive/rationality constraints to describe outcomes that are consistent with equilibrium behavior to some bargaining protocol. Retaining the standard notation, we denote the result of such bargaining by way of a direct bargaining mechanism which has two pieces: a probability of trade $p$ and transfer $x$ from the buyer to the seller. Because the investments are hidden actions, this bargaining is similar to the problem of bilateral trade with private information albeit here the initiation of bargaining is at an interim stage in some larger game in which private information in bargaining possibly arises from hidden actions at an earlier

[3]To be clear, the investment choice of player $i$ is unobservable to player $j$, but in equilibrium the players will correctly conjecture the other player's strategy. Furthermore, in any equilibrium in which $i$ employs a mixed strategy, she will be indifferent between all investment levels in the support of her mixture and weakly prefer these levels to investments not in the support of $F_i(\cdot)$.

stage of the game.

A direct bargaining mechanism is a game where the buyer and seller simultaneously report valuations, $v_i$ to a broker or mediator who then determines whether the object will be transferred, $p$, and at what price, $x$. We let the message space be the set of all valuations that can result from investment. Formally, a direct bargaining mechanism is defined by two mappings. The first $p(m_s, m_b) : \mathbb{R}^2_+ \to [0, 1]$ determines the probability of trade and the second, $x(m_s, m_b) : \mathbb{R}^2_+ \to \mathbb{R}$ describes the transfer from the buyer to seller. The total payoffs for a profile of messages and valuations are

$$W_s(v_s, m_s, m_b) = v_s(1 - p(m_s, m_b)) + x(m_s, m_b) - c_s(v_s)$$
$$W_b(v_b, m_b, m_s) = v_b p(m_s, m_b) - x(m_s, m_b) - c_b(v_b).$$

We will employ standard techniques to restrict consideration to direct bargaining mechanism that are Bayesian Incentive Compatible, i.e truth telling is a mutual best response to the mechanism given the investment lotteries employed. Our focus is on settings in which the players first make simultaneous investment decisions and correctly anticipate the direct bargaining mechanism.[4] Treating bargaining as an interim stage requires augmenting the concept of Bayesian Nash equilibrium to ensure that in determining what messages are best responses players use beliefs that are consistent with an equilibrium conjecture of the other players investment strategy and that in-

[4]Perhaps a more appropriate term would be "interim direct bargaining mechanism," but since we do not have any other mechanism, we will drop the qualified "interim".

vestment strategies are mutual best responses, given equilibrium conjectures about the reporting strategies. Given a direct bargaining mechanism, a strategy profile for the trading game is a pair of lotteries over investments and reports, where reports may depend on the realization of the possibly mixed investment actions. Thus, a strategy for player $i$ is a lottery $F_i(\cdot)$ with support $V_i \subset \mathbb{R}_+$ and a reporting rule $\sigma_i(v_i)$ that defines for every realization of a player's valuation what message she will send to the mechanism.

**Definition 1.** *An equilibrium is a direct bargaining mechanism $p(\cdot), x(\cdot)$, a pair of investment lotteries, $(F_s, F_b)$ and messaging strategies $(\sigma_b(v_b), \sigma_s(v_s))$ s.t. given the lotteries $(F_s, F_b)$, the messaging strategies constitute mutual best responses to direct bargaining mechanism $(p(\cdot), x(\cdot))$ (i.e. they are Bayesian incentive compatible) and, given the valuation contingent payoffs associated with play of the bargaining mechanism and messaging strategies, the investment strategies $(F_s, F_b)$ are simultaneous best responses. An equilibrium is truthful if the messaging strategies are the identity mapping, $\sigma_i(v_i) = v_i$.*

**Remark** : *Employing the logic in Myerson and Saterthwaite's proof of the revelation principle one can see that it is sufficient for us to focus on equilibria that are truthful. In the sequel we will focus only on truthful equilibria and for economy of exposition we suppress the adjective truthful, thus referring to equilibria to mean truthful equilibria.*

Most interesting trading games will also satisfy the condition that participation is voluntary and that the trading game maximizes social welfare.

In the bilateral trade setting with incomplete information Myerson and Satterthwaite (1983) restrict attention to games that satisfy an *interim* participation constraint where each type's expected net payoff from participating in the game is non-negative. In what follows, we will require that the equilibrium also satisfies this condition after investments are realized.

**Definition 2.** *An equilibrium to a trading game satisfies the interim participation constraint (Condition IP) if each player's expected gains from trade is non-negative for almost every valuation, $v_i \geq 0$.*

Second, we are interested in the relevance of time-consistency and precommitment to a bargaining mechanism or trading scheme that is optimal given rational expectations about investing behavior.

**Definition 3.** *We say that an equilibrium is interim optimal (Condition O) if, given the investment lotteries $(F_s, F_b)$, the bargaining mechanism $(p(\cdot), x(\cdot))$ maximizes the sum of players' payoffs within the class of mechanisms that are incentive compatible and which satisfy the interim participation constraint given the lotteries, $F_s(\cdot), F_b(\cdot)$.*

This model and notion of equilibrium captures two ideas: (1) That when making investment decisions the traders have rational expectations about how bargaining will unfold and (2) trade will be conducted in a manner that is second-best given equilibrium conjectures about investing strategies. One way of motivating this definition of equilibrium is to think of a game with three players: buyer, seller, and a broker who selects a direct bargaining mechanism after investments and who seeks to maximize the total utility to the buyer and seller.[5] This buyer would select the best of the direct bargaining protocals satsifying the relevant incentive constraints given correct beliefs about the mixed/behavioral investment strategies employed by the buyer and seller. Every equilibrium satisfying condition O would also be supported as a Bayesian Nash equilibrium in this three player game. Although the broker would clearly prefer equilibria in which she simply allocated the item to a particular trader and only that trader invested, every equilibrium satisfying condition O involves the choice of an optimal mechanism from the designer's perspective given equilibrium conjectures about the investing behavior and best responses by the buyer and seller given this bargaining mechanism. The converse is also true, any equilibrium to the 3 player game would also satisfy the conditions to be an equilibrium that also satisfies condition O. [6] A second motivation would be to conceive of a dynamic process where markets move toward efficient trading mechanisms. An equilibrium satisfying condition O can then be a steady-state to such a process.

[5]It is worth noting that Condition O can only hold on the path. Investments are hidden actions and thus if a player deviates from equilibrium the broker will not know this and cannot adjust and select the second-best given the distribution induced by the deviation.

[6]To clarify a potentially confusing aspect of condition O, we note that the optimality of the bargaining mechanism given buyer and seller (possibly mixed) investment strategies stems not from the latter reporting their choices to the designer, but from the designer correctly conjecturing the buyer's and seller's equilibrium mixtures. This means that the designer does not observe and therefore cannot react to deviations from the equilibrium strategy by the buyer or seller. This fact is crucial to our unraveling result.

# 3 Results

Let $F_i$ be player $i$'s mixed-strategy equilibrium distribution over the hidden action. Recall that our direct mechanism is a pair of functions $x(m_s, m_b)$ that describes the report-contingent transfer to the seller and a function $p(m_s, m_b)$ that determines the probability of trade. Expected gains from trade to the seller of reporting $m_s$ in this direct mechanism, given investment $v_s$, can then be written as the integral[7]

$$U_s(v_s', v_s) = \int_{V_b} [x(v_s', v_b) - p(v_s', v_b)v_s] dF_b(v_b) \, . \tag{1}$$

Similarly, for the buyer we have:

$$U_b(v_b', v_b) = \int_{V_s} [p(v_s, v_b')v_b - x(v_s, v_b')] dF_s(v_s). \tag{2}$$

In a slight abuse of notation, let $U_i(v_i) = U_i(v_i, v_i)$.

We note a convenient feature of the supports of investment strategies. Since $c_i(0) = 0$, if

$$c_i(\hat{v}_i) > \hat{v}_i \, ,$$

then the investment $\hat{v}_i$ is strictly dominated by $v_i = 0$. Recall that $\overline{\overline{v}}_i$ is the investment that makes $c_i'(v_i) = 1$ and so given strict convexity of the cost function any investment higher than this level yields a payoff strictly

[7] Throughout we denote Lebesgue-Stieltjes integrals with $dF_i(v_i)$ and Riemann integrals by $f_i(v_i)dv_i$-using the latter on intervals in which a density exists.

less than 0. In equilibrium investments must have support contained in the interval $[0, \overline{\overline{v}}_i]$. We can then conclude that equilibrium investment strategies always have compact support.

We now turn to the study of what types of investment strategies are possible in an equilibrium. We find that the equilibrium conditions from strategic investment pin down a number of characteristics of the bargaining problem.

**Theorem 1.** (Mixing theorem) *In any equilibrium, if $v_i$ is an accumulation point of the support of i's mixed strategy, then*

$$1 + U_s'(v_s) = c_s'(v_s), \tag{3}$$

$$U_b'(v_b) = c_b'(v_b). \tag{4}$$

The proof is given in the appendix. With investment in mixed strategies, it must be the case that for every point in the support of the investment actions either the derivative of the cost function and the utility are equal (if player 2) or differ by exactly 1 (if player 1). The derivative of the utility for the trading game is pinned down by incentive compatibility so there must be a close connection between investment strategies, their implied trading probabilities, and the marginal cost of investment for the traders. Below, we show that this connection precludes equilibria with investment decisions that lead to Myerson-Satterthwaite inefficiencies.

We start by considering the classical bilateral trading case investigated by Myerson and Satterthwaite, where both the buyer's and seller's valuations are distributed continuously over a connected domain. Myerson and Satterthwaite's classical result is that as long as the distributions of the players overlap and have full support on an interval, no efficient mechanism exists that is both incentive compatible and individually rational. The theorem below, on the other hand, shows that such distributions cannot emerge from a mixed-strategy investment equilibrium, if the mechanism designer is choosing second-best mechanisms that maximize aggregate gains from trade.

**Theorem 2.** (No Connected Supports with IC, O, IP) *Assume the cost function is strictly increasing. When the designer chooses an optimal IC and IP mechanism that maximizes aggregate gains from trade given the investment strategies (condition O) there is no mixed-strategy equilibrium with connected and overlapping supports containing no atoms.*

To see this, suppose the seller and the buyer are following mixed strategies with positive probability densities over $[a_s, b_s]$ and $[a_b, b_b]$, respectively, and that the interiors of the supports have a non-empty intersection. Myerson and Satterthwaite show that no efficient mechanism is possible under these assumptions. So, the aggregate gains from trade will be maximized by a second-best mechanism characterized by Theorem 2 of Myerson and Satterthwaite. This result states that the optimal second-best mechanism will ensure $U_s(b_s) = U_b(a_b) = 0$. This means that the lowest type buyer will not gain any benefit from their investment. It immediately follows that any

$a_b > 0$ is strictly dominated by investing 0 and not paying a cost; hence $a_b = 0$. Now, from Theorem 1 and the envelope theorem of Myerson and Satterthwaite we have for any incentive compatible mechanism:

$$c_b'(v_b) = \overline{p}_b(v_b) \tag{5}$$

Again, by Theorem 2 of Myerson and Satterthwaite, we know that the optimal second-best mechanism prescribes trade when

$$v_b - v_s \geq \alpha \left( \frac{F_s(v_s)}{f_s(v_s)} + \frac{1 - F_b(v_b)}{f_b(v_b)} \right) , \tag{6}$$

where $\alpha \in [0,1]$ and $F_i(\cdot)$ and $f_i(\cdot)$ are the cumulative and probability density functions for the players. Note that $\frac{1-F_b(0)}{f_b(0)} > 0$; hence, the right-hand side of (6) is strictly positive. This means that there exists an $\epsilon > 0$, for which $\frac{1-F_b(\epsilon)}{f_b(\epsilon)} > \epsilon$; and consequently, $\overline{p}_b(\epsilon) = 0$. In other words, even if the lower bound of the seller's mixed strategy is at 0, the $\epsilon$-type buyer will not be able to trade with any seller because the IC and interim participation (IP) constraints mean that the $\epsilon$ type will not trade even with the 0-type seller. However, since $c_b'(\epsilon) > 0$ by assumption, it follows that the buyer strictly prefers a lower investment to the $\epsilon$-investment. This means that $\epsilon$ cannot be part of the equilibrium support, a contradiction.

The above result shows that "nice" lotteries over valuations and the inefficiencies from Myerson-Satterthwaite cannot arise in equilibria when valuations emerge from hidden investments. The reason is the wedge introduced by the IP conditions in the second-best bargaining mechanism, which ensure that the lowest type buyers cannot trade with anyone. But then these types cannot be supported by equilibrium investment decisions.

However, Theorem 2 above does not rule out potential investment strategies that involve both atoms and gaps. For example, one might think that placing a probability mass of sellers at zero investment, and a gap between the zero-type buyers and the next highest type in the mixed strategy's support might resolve the issue identified in the previous section. Therefore, we next consider this possibility by extending the Myerson-Satterthwaite analysis to distributions with gaps and atoms. We show that this extension might allow efficient mechanisms satisfying IC and IP in some cases. However, we also show that distributions that do not admit first-best efficiency in the trading stage cannot be equilibrium mixed strategies. As a result, the conclusion that Myerson-Satterthwaite inefficiencies cannot occur with endogenously determined investments extends beyond the case of lotteries described in Theorem 2. This result holds generally when the valuations are equilibrium choices as modeled here.

To show this result, consider a distribution whose support consists of an arbitrary (but countable) number of atoms, gaps, and compact intervals. Figure 1 illustrates a simple example of such a distribution where the atoms are at the upper and lower limits of the distribution.[8] Denote by $\Theta_i$ the

[8] The following result shows that unraveling occurs if there are gaps in combination with atoms. We can rule out the possibility of atoms in the interior of either player's support with standard arguments on all-pay auctions (e.g. Baye, Kovenock and De Vries, 1996)

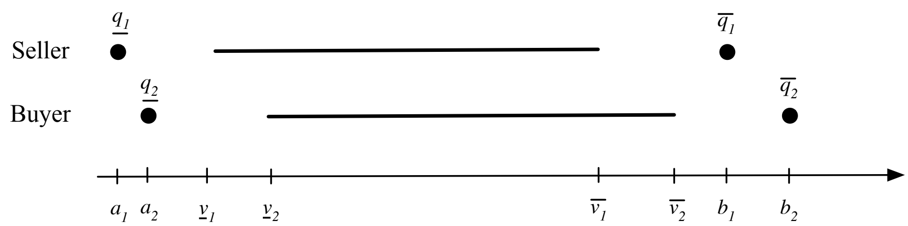


Figure 1: Diagram of the buyer and seller distributions. Solid lines signify continuous supports of the players' investment strategies.

set of atoms and by $\mathcal{K}_i$ the union of all compact intervals in the support of player $i$'s mixed strategy distribution. We use $\mathcal{I}_i^j$ to denote the $j$th such interval (counted in increasing order). Likewise, for notational convenience, we define the sets $\underline{V}_i$ and $\overline{V}_i$ to be the sets of infima and suprema of the compact intervals in the distribution of the seller and the buyer, respectively, with $\underline{v}_i^j = \inf(\mathcal{I}_i^j)$ and $\overline{v}_i^j = \sup(\mathcal{I}_i^j)$.

To focus on efficient mechanisms, we make the technical assumption that $p(v_s, v_b)$ is left-continuous in $v_s$ and right-continuous on $v_b$, which is satisfied for both efficient mechanisms and second-best mechanisms because any discontinuity in such mechanisms will involve $v_b \geq v_s$, and the mechanism designer will weakly prefer trade to non-trade. Given a mechanism with allocation function $p(v_s, v_b)$ and transfer function $x(v_s, v_b)$, we define the expected probability of trading for a seller that reports her type as $v_s$, $\overline{p}_s(v_s)$,

and the fact that an optimal M-S mechanism induces a probability of trade equalling 0 or 1.

with the Lebesque-Stieltjes integral:

$$\overline{p}_s(v_s) = \int_{a_b}^{b_b} p(v_s, t_b) dF_b \ , \tag{7}$$

where $a_b$ and $b_b$ are again the lower and upper bounds of the buyer's distribution. Under our assumptions $\overline{p}_s$ is left-continuous (and $\overline{p}_b$ right-continuous). Likewise, the expected payment to the seller reporting $v_s$ is defined as

$$\overline{x}_s(v_s) = \int_{a_b}^{b_b} x(v_s, t_b) dF_b \ . \tag{8}$$

The expected gain (relative to non-participation) for the seller from declaring $v_s'$ when his real type is $v_s$, is:

$$U_s(v_s', v_s) = \int_{\underline{v}_b}^{\overline{v}_b} [x(v_s', v_b) - p(v_s', v_b) v_s] dF_b \ . \tag{9}$$

Similar definitions apply to the buyer. These integrals exist because $p(\cdot,\cdot)$ and $x(\cdot,\cdot)$ are non-negative and $F_i$ are monotone and right-continuous. First, we show that the envelope theorem applies in the connected parts of the seller and buyer's distribution, and put bounds on the difference between the expected payoffs for types bordering the gaps. For brevity, the proofs of the following results are given in the Appendix.

**Lemma 1.** Envelope theorem with atoms and gaps *Consider an IC mechanism. For any $v_s \in \mathcal{K}_s$, the expected payoff satisfies $U_s'(v_s) = -\overline{p}_s(v_s).$,* *and if* $v_s \in \mathcal{I}_s^j$ *then the following holds:*

$$U_s(v_s, v_s) = U_s(\overline{v}_s^j, \overline{v}_s^j) + \int_{v_s}^{\overline{v}_s^j} \overline{p}_s(t_s) dt_s \ . \tag{10}$$

*Likewise, for any* $v_b \in \mathcal{K}_b$*, the expected payoff satisfies* $U_b'(v_b) = \overline{p}_b(v_b)$ *and for* $v_b \in \mathcal{I}_b^j$*:*

$$U_b(v_b, v_b) = U_b(\underline{v}_b^j, \underline{v}_b^j) + \int_{\underline{v}_b^j}^{v_b} \overline{p}_b(t_b) dt_b \ . \tag{11}$$

*For two values* $v_s$ *and* $v_s'$ *that border a gap in the seller's distribution, we have:*

$$-\overline{p}_s(v_s') \geq \frac{U_s(v_s) - U_s(v_s')}{v_s' - v_s} \geq -\overline{p}_s(v_s) \ , \tag{12}$$

*and likewise, for two values* $v_b$ *and* $v_b'$ *that border a gap in the buyer's distribution, we have:*

$$\overline{p}_b(v_b') \geq \frac{U_b(v_b') - U_b(v_b)}{v_b' - v_b} \geq \overline{p}_b(v_b) \ . \tag{13}$$

For the next theorem which gives the necessary and sufficient condition for an IP and IC mechanism to satisfy, let us denote by $\text{supp}(F_i)$ the support of the distribution of player $i$, and the functions $\overline{\pi}_s(v_s)$ and $\overline{\pi}_b(v_b)$, as follows:

$$\overline{\pi}_s(v_s) = \begin{cases} \overline{p}_s(v_s) & \text{if } v_s \in \operatorname{supp}(F_s) \\ \overline{p}_s(\hat{v}_s) & \text{s.t. } \hat{v}_s = \inf\{x \in \operatorname{supp}(F_s) \,|\, x \geq v_s\} \text{ otherwise,} \end{cases} \tag{14}$$

$$\overline{\pi}_b(v_b) = \begin{cases} \overline{p}_b(v_b) & \text{if } v_b \in \operatorname{supp}(F_b) \\ \overline{p}_b(\hat{v}_b) & \text{s.t. } \hat{v}_b = \sup\{x \in \operatorname{supp}(F_b) \,|\, x \leq v_b\} \text{ otherwise.} \end{cases} \tag{15}$$

In other words, $\overline{\pi}_s(v_s)$ is equal to $\overline{p}_s(v_s)$ whenever $v_s$ is in the support of the seller's distribution, and equal to $\overline{p}_s$ for the next higher point in the seller's distribution, if $v_s$ is not in the support. Likewise for the buyer, except if $v_b$ is not in the support of the buyer's distribution, $\overline{\pi}_b(v_b)$ is equal to the expected probability of trade for the next lower point in the distribution.

**Theorem 3.** (Myerson & Satterthwaite with atoms and gaps). *Given buyer and seller distributions $F_s$ and $F_b$, consisting of a countable number of atoms and compact supports given by the union of intervals, for any IC and IP mechanism $(x, p)$ it must hold that:*

$$\begin{aligned} &\int_{a_b}^{b_b} v_b \overline{p}(v_b) dF_b - \int_{a_s}^{b_s} v_s \overline{p}(v_s) dF_s \\ &\qquad - \int_{a_s}^{b_s} F_s(t_s) \overline{\pi}_s(t_s) dt_s - \int_{a_b}^{b_b} (1 - F_b(t_b)) \overline{\pi}_b(t_b) dt_b \\ &\qquad\qquad \geq U_s(b_s) + U_b(a_b) \geq 0\,. \end{aligned} \tag{16}$$

*Furthermore, for any function $p(v_s, v_b)$ that maps from* $\operatorname{supp}(F_s) \times \operatorname{supp}(\mathrm{F_b})$

*to* $[0,1]$*, a payment function* $x(v_s, v_b)$ *exists such that* $(x, p)$ *is IC and IP if and only if* (16) *holds and* $\overline{p}_s(v_s)$ *and* $\overline{p}_b(v_b)$ *are weakly decreasing and decreasing, respectively.*

With Theorem 3, we have the following Lemma:

**Lemma 2.** *Efficient trading mechanisms that satisfy the IC and IP conditions exist for some distributions of buyers' and sellers' valuations when atoms and gaps are allowed.*

To prove this lemma, we only need to find an example of distributions that satisfy the constraint (16). Suppose the distributions of the buyers and sellers are of the sort depicted in Figure 1, i.e., both distributions have atoms at their extreme values, and two gaps separating these atoms from the connected part of the support. For a set of values, we can easily evaluate the inequality (16) numerically, assuming efficient trade (i.e., $\overline{p}_s(v_s) = 1 - F_b(v_s)$ and $\overline{p}_b(v_b) = F_s(v_b)$). Assume the following values (see Figure 1): $b_b = 1$, $b_s = 0.7$, $\overline{v}_b = 0.65$, $\overline{v}_s = 0.6$, $\underline{v}_b = 0.4$, $\underline{v}_s = 0.2$, $\underline{q}_s = 0.1$, $\overline{q}_s = 0.6$, $\underline{q}_b = 0.1$, and $\overline{q}_b = 0.6$. With these values the left-hand side of (16) can be numerically evaluated, and is found to be $0.0079 > 0$, even though the distributions have overlapping support.

This example shows the Myerson-Satterthwaite Theorem fails when atoms and gaps are allowed in the distribution of valuations. The example also complements an example provided by Myerson and Satterthwaite (1983, p.273) with overlapping support of the buyer and seller's distribution of valuations and a probability function with atoms only on the end points of the distribution. The failure of the impossibility result requires not just atoms but atoms and gaps. Clearly, characterizing these conditions is crucial for deriving results with mixed strategies that may lead to distributions of player valuations with atoms and gaps. As a side benefit, the lemma clarifies why distributions without full support can lead to efficient bargaining outcomes.

Why is it possible to support efficiency with overlapping supports when atoms and gaps exist? Even with more general distributions of valuations the envelope theorem dictates how the probability of trade must change with valuation levels. The presence of gaps creates slack for some types of the other player, because the relevant deviations for these types lead to discontinuous jumps in their utility. When the type that has slack is an atom then this slack can effectively yield a subsidy that can be transferred to trades whose valuation comes from the connected part of the support. The higher the mass on the atoms the more subsidy is available to the designer to redistribute to the rest of the types. We can then think of the designer as using the subsidy from the atoms like a broker uses a subsidy, to create incentives for truth-telling on the overlapping regions of the support. Therefore, efficiency tends to be possible either when most of the probability weight is assigned to the atoms ($\overline{q}_i$ and $\underline{q}_i$ above), leaving little probability density in the overlapping parts of the distributions, or when the gaps between the connect range of values and the atoms are large. Both make it easier to incentivize truth-telling.

When it is impossible to implement efficient incentive compatible individually rational mechanism, we have the following results about the second-best mechanisms:

**Lemma 3.** *Consider any gap in the seller's distribution, and denote the lower and upper boundary of the gap as* $\underline{v}_s$ *and* $\overline{v}_s$*, respectively (i.e.,* $\underline{v}_s, \overline{v}_s \in \operatorname{supp}(F_s)$ *but for* $\underline{v}_s < t_s < \overline{v}_s$*,* $t_s \notin \operatorname{supp}(F_s)$*). When an efficient mechanism does not satisfy both IC and IP, the second-best mechanism maximizing aggregate gains from trade has:*

$$U_s(\underline{v}_s) = U_s(\overline{v}_s) + (\overline{v}_s - \underline{v}_s)\overline{p}_s(\overline{v}_s)$$

*Similarly, consider any gap in the buyer's distribution, bounded by* $\underline{v}_b$ *and* $\overline{v}_b$*. A second-best mechanism has*

$$U_b(\overline{v}_b) = U_b(\underline{v}_b) + (\overline{v}_b - \underline{v}_b)\overline{p}_b(\underline{v}_b)$$

Using Lemma 3, we can now prove that a second-best mechanism when efficiency is not possible rules out a mixing investment equilibrium.

**Lemma 4.** (Mixed investment unravels) *Suppose that given* $F_s, F_b$ *with atoms and gaps there is no IC and IP mechanism that is efficient and the designer chooses a second-best mechanism* $(p, x)$ *to maximize the aggregate gains from trade given these lotteries (condition O). Then it is not possible to support* $F_s, F_b$ *as equilibrium mixed investment strategies with* $(p, x)$ *for* *any strictly convex and continuous cost functions.*

Theorem 4 follows directly from Lemmas (2) and (4).

**Theorem 4.** *There is no equilibrium satisfying conditions IP and O with the form of inefficiency that appears in Myerson-Satterthwaite. In otherwords, every equilibrium satisfying conditions IP and O allocates the good to the player with a higher valuation with probability one.*

## 3.1 Allocation inefficiency

The results above show that with pre-trade investment, there is no equilibrium of a trading game that has the inefficiency identified by Myerson and Satterthwaite, and inefficient mixed investment strategies will unravel. However, even if the agent with the item invests optimally with probability 1, a third form of inefficiency might persist, in that the agent who ends up with the item might not be the one capable of producing the highest surplus. An important question is then, in bilateral trade with investments, are there failures of Coasian efficiency? That is, do there exist initial allocations of the good such that the individual who extracts less utility from ownership maintains or obtains the property right to the good in equilibrium?

We maintain the assumption that the good is initially assigned to one player (the seller). In principle, a larger class of schemes involving a market maker or auctioneer can be considered. We will say a player is the efficient owner if she is the player who is able to generate the highest utility from the use of the good. In practice, such a user would have a cost function that allows for lower production costs. First, suppose that the seller, who initially owns the good, is the efficient user. In this situation allocative efficiency always occurs in equilibrium. If such a seller keeps the item with probability 1, then in equilibrium, he will choose an investment level to maximize his utility of ownership. In any equilibrium in which the buyer gets the good with probability 0, her investment must be equal to 0.

To see that nothing else can be supported in an equilibrium, suppose the seller trades the good. At most the buyer pays an amount

$$x^* = \max_{v_b} \left( v_b - c_b(v_b) \right).$$

However, we have assumed for this case that

$$\max_{v_b} \left( v_b - c_b(v_b) \right) < \max_{v_s} \left( v_s - c_s(v_s) \right)$$

and the seller is better off not participating in the trade, keeping the good, and investing optimally. Importantly, the buyer is never willing to pay a price that makes the seller willing to sell and there is no equilibrium where the seller trades the good when he is the efficient owner.

When the buyer is the efficient owner of the good, i.e., when

$$\max_{v_b} \left( v_b - c_b(v_b) \right) > \max_{v_s} \left( v_s - c_s(v_s) \right) \ ,$$

the picture is more complicated and there are cases where there are equilibria in which (i) only the seller (less efficient type) invests and (ii) the probability of trade is 0. This happens under two conditions. The first one is when the optimal investment by the seller is greater than the surplus generated by the buyer's optimal investment, i.e., when:

$$\arg\max_{v_s}\left(v_s - c_s(v_s)\right) > \max_{v_b}\left(v_b - c_b(v_b)\right)\ .$$

When this condition holds there is no price that the buyer can pay to an optimally investing seller that would induce trade and result in a positive total utility to the buyer. Hence, the seller investing optimally and the buyer not investing occurs in an equilibrium. However, the seller not investing, the buyer investing optimally, and trade taking place, is also an equilibrium provided that the price $p$ satisfies

$$\max_{v_b}\left(v_b - c_b(v_b)\right) > p > \max_{v_s}\left(v_s - c_s(v_s)\right)\ .$$

The second case that admits an inefficient equilibrium stems from perhaps rather peculiar cost functions. In particular, suppose again the buyer is the efficient owner, but that

$$\arg\max_{v_s}\left(v_s - c_s(v_s)\right) > \arg\max_{v_b}\left(v_b - c_b(v_b)\right)\ .$$

In other words, even though the buyer can generate more surplus, she does this with a lower absolute investment level than the seller's maximum surplus. In this case, again, trade between efficiently invested sellers and buyers is impossible. Hence, the buyer investing efficiently and the seller not investing occurs in an equilibrium. Furthermore, as in the previous case, the efficient outcome (seller not investing, buyer investing efficiently, and trading) is also supportable in an equilibrium, as long as the price $p$ satisfies the same inequalities above.

Note that in both cases the inefficient equilibrium could be dispensed with if, prior to the investment stage, the buyer can sell an option to the seller that commits the buyer to buy at a price within the range identified with these inequalities. In this case, the seller can always ensure a higher payoff by not investing and selling at that price instead of investing optimally and not trading - regardless of the buyer's actions. Accordingly, in equilibrium, the seller will invest 0 and sell the item. The buyer's best response is to invest optimally. Hence, only the efficient equilibrium remains.

To recapitulate, if the initial allocation of the good is to the inefficient player (one who can generate less surplus), then allowing trade after unobservable investments does not necessarily guarantee that the best allocation will be achieved. However, this result does not stem from private information during the trading phase. In fact, all equilibria discussed above imply that players will follow pure investment strategies, the inefficiencies result from the inability of the parties to trade at a high enough price. If either a designer or the buyer can commit to trade at a high enough price, the inefficient equilibria can be made to go away. Even though our results do not guarantee first-best outcomes in the sense that the player who can make the most out of owning the good possesses it in all equilibria, we find that informational asymmetries are not the cause of the inefficiencies that remain at equilibrium.

## 4 Related Literature

As noted above Gul (2001) shows that if the buyer's investment is a hidden action, then, even when the seller has all the bargaining power, the underinvestment problem can be resolved if repeated offers are allowed and the time between offers vanishes. Gul also considers the case of two-sided investments but assumes that the seller's investment is observed prior to bargaining.[9] Gul's model only allow asymmetric information to emerge from strategic uncertainty caused by mixed strategies and hidden actions. We share this feature but allow for hidden actions by both players.

In the other papers on pre-bargaining investment four central distinctions appear. In some of this scholarship the relevant fundamentals, like investment in value, are assumed to be observable at the time of bargaining (Grossman and Hart, 1986; Hart and Moore, 1988). More recently, Schmitz (2002), González (2004) and Lau (2008) also consider investments as hidden actions, however, these papers consider the case where only one player can invest

[9]Incidentally, Gul finds that the seller will have an incentive to underinvest, and points out the challenges to applying his arguments to the case of a continuum of types.

and Lau (2008) allows for partial observability of investments. Lau (2011) considers the case of one-sided hidden investments and exogenous asymmetric information, capturing some of the relevant tradeoffs but in her paper the asymmetric information is not directly attributed to a choice by the players.

Perhaps closer to our paper is Rogerson (1992) who provides a quite general treatment of the case where multiple players can invest before trade and where there are no externalities. His case of completely private information is closest in spirit to the environments we consider. The key distinction is that Rogerson assumes that there is a random component connecting each player's investment to its type. In particular, by assuming that investment decisions always admit unique optima, he excludes the case where investments completely determine a player's type (as in Gul (2001) and our paper). Rogerson also does not impose the individual rationality constraint imposed by Myerson-Satterthwaite and thus, in principle, is free to work with a larger set of mechanisms (he does require budget balance and incentive compatibility). Finally, Rogerson assumes that the mechanism is committed to prior to investment decisions and shows that d'Aspremont and Gérard-Varet (1979) and Cremer and Riordan (1985) mechanisms also create incentives for optimal investment.[10] We are interested in the same participation constraints as Myerson and Satterthwaite, and focus only on mechanisms that are optimal given equilibrium beliefs about investments. Thus, we do not analyze the

[10]See also Hart and Moore (1988) for a similar observation in case of two-players and an indivisible item–as in our model.

full-mechanism design problem in which a designer commits to a mechanism (either before or after learning something from the traders) that is not the constrained optimal.

As Gul (2001) notes, a common feature of his result and the literature on moral hazard and renegotiation (Che and Chung, 1999; Che and Hausch, 1999; Fudenberg and Tirole, 1990; Hermalin and Katz, 1991; Ma, 1991, 1994; Matthews, 1995) is that pure strategies by the agent generate strong reactions from the principal and thus, in equilibrium, the agents' randomization generates asymmetric information. Our analysis offers a counter-point to this result. The presence of randomization by both traders is typically hard to support, and impossible to support when they admit no first-best trading mechanisms, and if the traders anticipate that a designer is using a second-best mechanism. One possible exception to this result occurs if the investment cost functions support an equilibrium in which the buyer and seller mix over a small interval and disconnected atoms, with most of the probability being allocated to the atoms. In cases like this first best trading rules exist, even though there is overlap in the supports. Therefore, this form of strategic uncertainty is not consistent with the inefficiency that emerges in Myerson and Satterthwaite, as first-best becomes possible with this information environment.

# 5 Conclusion

Sometimes the value of a trade between two economic agents is determined by choices that the traders make prior to the transaction. In these circumstances a rational expectation about how the trading game might be played can be seen to have important effects on the incentives to invest and, as a consequence, generate bilateral trade games where the information environment looks very different from those well-studied in the economics literature. When valuations are the product of hidden pre-trade investment, the standard connected set of types cannot emerge as the result of equilibrium mixing. Furthermore, in every equilibrium of the trading game, given investments, trade occurs in every instance where the net gains are positive. In short, this means that the Myerson-Satterthwaite inefficiencies do not arise in these environments.

We see that hidden investments by two players is qualitatively different than one-sided hidden actions in one fundamental way. In the one-sided case the hold-up problem is somewhat dependent on the presence of inefficiencies in bargaining. With two sided investments these forms of inefficiency are not complimentary. The possibility of the inefficiency in bargaining destroys the possibility of inefficiencies in investments.

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# 6 Appendix

*Proof.* **Theorem 1**. Consider an equilibrium involving the direct mechanism $(x, p)$. To begin, consider the case of the seller. Take any two investments $v_s, v'_s$ in the support of $F_s$. Then, because the seller is mixing over these values

$$\int_{V_b} (1 - p(v_s, v_b))v_s + x(v_s, v_b) dF_b(v_b) - c_s(v_s) =$$
$$\int_{V_b} (1 - p(v'_s, v_b))v'_s + x(v'_s, v_b) dF_b(v_b) - c_s(v'_s) \, .$$

The left-hand side equals

$$U_s(v_s) - c_s(v_s) + v_s,$$

and the right-hand side equals

$$U_s(v'_s) - c_s(v'_s) + v'_s,$$

and we can rewrite the equation above as

$$U_s(v_s) - c_s(v_s) + v_s = U_s(v'_s) - c_s(v'_s) + v'_s \tag{17}$$

$$1 + \frac{U_s(v_s) - U_s(v'_s)}{v_s - v'_s} = \frac{c_s(v_s) - c_s(v'_s)}{v_s - v'_s}. \tag{18}$$

At an accumulation point of the support of $F_s$, we can take the limits as

$v_i' \to v_i$ and

$$1 + U_s'(v_s) = c_s'(v_s). \tag{19}$$

This is the first equation in the theorem. Similar calculations give the identity for the seller. □

*Proof.* **Proof of Lemma 1** The proof follows the familiar argument of Myerson and Satterthwaite. Incentive compatibility means that for all $v_s$, $v_s'$ in the support of the seller's distribution:

$$U_s(v_s, v_s) \geq U_s(v_s', v_s) \tag{20}$$

$$U_s(v_s', v_s') \geq U_s(v_s, v_s')\ . \tag{21}$$

By subtracting the RHS of the second inequality from the LHS of the first and the RHS of the first from the second and canceling the payment terms, we get:

$$-\overline{p}_s(v_s)[v_s - v_s'] \geq U_s(v_s, v_s) - U_s(v_s', v_s') \geq -\overline{p}_s(v_s')[v_s - v_s']\ .$$

For either $v_s$ or $v_s'$ in $\Theta_s$, we can stop here. For $v_s \in \mathcal{K}_s$ and $v_s \notin \underline{V}_s$, we assume $v_s > v_s'$, divide by $v_s - v_s'$ and take the limit as $v_s' \to v_s$ to obtain:

$$U_s'(v_s) = -\overline{p}_s(v_s)\ . \tag{22}$$

For $v_s \in \underline{V}_s$ we simply take the limit from the right, with $v_s' > v_s$ to obtain the same result. Integrating equation (22) within an interval $\mathcal{I}_s^j$, we obtain (10) The same method applies to the buyers. □

*Proof.* **Proof of Theorem 3**

The proof proceeds analogously to the canonical case (theorem 1 of Myerson and Satterthwaite), except that we need to make use of Lebesgue-Stieltjes integrals to account for the fact that we integrate over distributions that have gaps and atoms. First, observe that Lemma 1 implies that $U_s(b_s) \leq U_s(v_s)$ for all $v_s$ in the seller's support, and $U_b(a_b) \leq U_b(v_b)$ for all $v_b$ in the buyer's support. Next, consider the expected gains from trade under a direct mechanism $(x, p)$.

$$\begin{aligned}\int_{a_s}^{b_s} \int_{a_b}^{b_b} (v_b - v_s) p(v_s, v_b) dF_b dF_s &= \int_{a_s}^{b_s} \int_{a_b}^{b_b} v_b p(v_s, v_b) dF_b dF_s \\ &\quad - \int_{a_s}^{b_s} \int_{a_b}^{b_b} -v_s p(v_s, v_b) dF_b dF_s \\ &= \int_{a_b}^{b_b} v_b \overline{p}(v_b) dF_b - \int_{a_s}^{b_s} v_s \overline{p}(v_s) dF_s \,, \end{aligned} \tag{23}$$

where the last line follows from integrating the two integrals in different orders, permissible by Tolleni's theorem.

At the same time, since the payments are zero sum, the expected gains from trade is equal to the sum of the average gains of the buyers and sellers:

$$\int_{a_s}^{b_s} \int_{a_b}^{b_b} (v_b - v_s) p(v_s, v_b) dF_b dF_s = \int_{a_s}^{b_s} U_s(v_s) dF_s + \int_{a_b}^{b_b} U_b(v_b) dF_b \tag{24}$$

Take the seller's term, the first integral. Using the envelope theorem (Lemma 1) and using the definition of the function $\overline{\pi}_s(v_s)$ above, we can write

$$U_s(v_s) \geq U_s(b_s) + \int_{v_s}^{b_s} \overline{\pi}_s(t_s)dt_s \tag{25}$$

So, we have:

$$\begin{aligned}\int_{a_s}^{b_s} U_s(v_s)dF_s &\geq \int_{a_s}^{b_s} \left[U_s(b_s) + \int_{v_s}^{b_s} \overline{\pi}_s(t_s)dt_s\right] dF_s \\ &= U_s(b_s) + \int_{a_s}^{b_s}\int_{v_s}^{b_s} \overline{\pi}_s(t_s)dt_s dF_s = U_s(b_s) + \int_{a_s}^{b_s} F_s(t_s)\overline{\pi}_s(t_s)dt_s \ ,\end{aligned}$$

where the change in the order of integration again is permissible by Tolleni's theorem. Similarly for the buyer, we have:

$$\begin{aligned}\int_{a_b}^{b_b} U_b(v_b)dF_b &\geq \int_{a_b}^{b_b} \left[U_b(a_b) + \int_{a_b}^{t_b} \overline{\pi}_b(t_b)dt_b\right] dF_b \\ &= U_b(a_b) + \int_{a_b}^{b_b} (1 - F_b(t_b))\overline{\pi}_b(t_b)dt_b \ .\end{aligned}$$

Putting these together, we have:

$$\begin{aligned}\int_{a_b}^{b_b} v_b\overline{p}(v_b)dF_b - \int_{a_s}^{b_s} v_s\overline{p}(v_s)dF_s &\geq U_s(b_s) + U_b(a_b) + \\ &\int_{a_s}^{b_s} F_s(t_s)\overline{\pi}_s(t_s)dt_s + \int_{a_b}^{b_b} (1 - F_b(t_b))\overline{\pi}_b(t_b)dt_b \ , \end{aligned} \tag{26}$$

or,

$$\int_{a_b}^{b_b} v_b \overline{p}(v_b) dF_b - \int_{a_s}^{b_s} v_s \overline{p}(v_s) dF_s$$
$$- \int_{a_s}^{b_s} F_s(t_s)\overline{\pi}_s(t_s) dt_s - \int_{a_b}^{b_b} (1 - F_b(t_b))\overline{\pi}_b(t_b) dt_b \geq U_s(b_s) + U_b(a_b) \geq 0\,. \tag{27}$$

This proves the "only if" part of theorem 3. To prove the "if" part, we need to show that for a function $p(\cdot,\cdot)$ satisfying (16), and when $\overline{p}_s(\cdot)$ and $\overline{p}_b(\cdot)$ are weakly decreasing and increasing, respectively, a payment function exists that makes the mechanism satisfy IC and IP. First, we observe that for $\overline{p}_s(\cdot)$ and $\overline{p}_b(\cdot)$ are weakly decreasing and increasing, respectively, $\overline{\pi}_s(\cdot)$ and $\overline{\pi}_b(\cdot)$, defined in (14) and (15) are also weakly decreasing and increasing, respectively.

Next, consider the following payment function:

$$x(v_s, v_b) = \chi_b(v_b) - \chi_s(v_s) + K\,, \tag{28}$$

where $\chi_s(\cdot)$ and $\chi_b(\cdot)$ are given by the Lebesgue-Stieltjes integrals:

$$\chi_b(v_b) = \int_{t_b=a_b}^{v_b} t_b d[\overline{\pi}_b(t_b)] \tag{29}$$
$$\chi_s(v_s) = \int_{t_s=a_s}^{v_s} t_s d[-\overline{\pi}_s(t_s)] \tag{30}$$

and $K$ is a constant. To see that this payment function satisfies incentive compatibility, consider for any pair $v_s$, $v_s'$ in the seller's support:

$$U_s(v_s, v_s) - U_s(v_s', v_s) = -v_s(\overline{p}_s(v_s) - \overline{p}_s(v_s')) - \chi_s(v_s) + \chi_s(v_s')$$

Since $\overline{\pi}_s(v_s) = \overline{p}_s(v_s)$ whenever $v_s$ is in the support of the seller, we have $\overline{p}_s(v_s) - \overline{p}_s(v_s') = -\int_{t_s=v_s'}^{v_s} d[-\overline{\pi}_s(t_s)]$, and $-\chi_s(v_s) + \chi_s(v_s') = -\int_{t_s=v_s'}^{v_s} t_s d[-\overline{\pi}_s(t_s)]$ thus we have:

$$\begin{aligned} U_s(v_s, v_s) - U_s(v_s', v_s) =& v_s \int_{t_s=v_s'}^{v_s} d[-\overline{\pi}_s(t_s)] - \int_{t_s=v_s'}^{v_s} t_s d[-\overline{\pi}_s(t_s)] \\ =& \int_{t_s=v_s'}^{v_s} (v_s - t_s) d[-\overline{\pi}_s(t_s)] \geq 0 \ , \end{aligned} \tag{31}$$

since $\overline{\pi}_s(\cdot)$ is a weakly decreasing function. The proof for the buyer proceeds analogously.

Now, consider the difference $U_s(v_s') - U_s(v_s)$ for some $v_s' \leq v_s$ in the seller's support:

$$\begin{aligned} U_s(v_s') - U_s(v_s) =& - v_s'\overline{p}_s(v_s') + v_s\overline{p}_s(v_s) - \chi_s(v_s') + \chi_s(v_s) \\ =& - v_s'\overline{p}_s(v_s') + v_s\overline{p}_s(v_s) + \int_{t_s=v_s'}^{v_s} t_s d[-\overline{\pi}_s(t_s)] \\ =& - v_s'\overline{p}_s(v_s') + v_s\overline{p}_s(v_s) + \int_{t_s=v_s'}^{v_s} \overline{\pi}_s(t_s) dt_s - \Big[t_s\overline{\pi}_s(t_s)\Big]_{t_s=v_s'}^{v_s} \\ =& \int_{t_s=v_s'}^{v_s} \overline{\pi}_s(t_s) dt_s \ , \end{aligned} \tag{32}$$

where the second to last step follows from integration by parts (we note that

$\overline{p}_s$ is left-continuous and non-increasing under our assumptions), and the last step is due to the fact that $\overline{\pi}_s(v_s) = \overline{p}_s(v_s)$ by definition whenever $v_s$ is in the support of the seller. Thus, the payment function (28) yields for any $v_s$ in the seller's support:

$$U_s(v_s) = U_s(b_s) + \int_{t_s=v_s}^{b_s} \overline{\pi}_s(t_s)dt_s \tag{33}$$

A similar calculation shows that for any $v_b$ in the buyer's support, we have:

$$U_b(v_b) = U_b(a_b) + \int_{t_b=a_b}^{v_b} \overline{\pi}_b(t_b)dt_b \tag{34}$$

These two relations imply that under this payment function, the inequality in (25) (and the corresponding one for the buyer) is satisfied with equality, and through the steps that follow, the first inequality in (16) must also be satisfied with equality, and that if the LHS of it is non-negative, $U_s(b_s) + U_b(a_b)$ must also be non-negative.

Now consider $U_s(b_s)$. We have

$$\begin{aligned} U_s(b_s) &= \int_{a_b}^{b_b} (x(b_s, v_b) - b_s p(b_s, v_b))dF_b \\ &= \int_{a_b}^{b_b} \int_{t_b=a_b}^{v_b} t_b d[\overline{\pi}_b(t_b)]dF_b - \int_{t_s=a_s}^{b_s} t_s d[-\overline{\pi}_s(t_s)] + K - b_s\overline{p}_s(b_s) \\ &= \int_{t_b=a_b}^{b_b} (1 - F_b(t_b))t_b d[\overline{\pi}_b(t_b)] - \int_{t_s=a_s}^{b_s} t_s d[-\overline{\pi}_s(t_s)] - b_s\overline{p}_s(b_s) + K \end{aligned} \tag{35}$$

Setting

$$K = -\int_{t_b=a_b}^{b_b} (1 - F_b(t_b))t_b d[\overline{\pi}_b(t_b)] + \int_{t_s=a_s}^{b_s} t_s d[-\overline{\pi}_s(t_s)] + b_s\overline{p}_s(b_s) \quad (36)$$

ensures that $U_s(b_s) = 0$. Since, in addition, we assume that the LHS of (16) is non-negative, and have shown that it is equal to $U_s(b_s) + U_b(a_b)$, it must follow that $U_b(a_b)$ is also non-negative. This implies, by the envelope theorem, that the mechanism is IP for all buyer and seller types. □

*Proof.* **Proof of lemma 3**

We will prove the lemma for the case of the seller; the proof works exactly the same way for the buyer.

For a certain trading probability function $p(v_s, v_b)$, that yields non-increasing and non-decreasing $\overline{p}_s(\cdot)$ and $\overline{p}_b(\cdot)$, respectively, consider a mechanism resulting in the following relationship at the focal gap in the seller's distribution:

$$U_s(\underline{v}_s) = U_s(\overline{v}_s) + (\overline{v}_s - \underline{v}_s)(\overline{p}_s(\overline{v}_s) + \gamma)$$

where $0 \leq \gamma \leq (\overline{p}_s(\overline{v}_s) - \overline{p}_s(\underline{v}_s))$, such that incentive compatibility is satisfied for the sellers of type $\overline{v}_s$ and $\underline{v}_s$. Under this mechanism, for all $v_s \leq \underline{v}_s$, the envelope theorem will have an additional payoff increment $\gamma(\overline{v}_s - \underline{v}_s)$ for all

$v_s \leq \underline{v}_s$, so we need to modify inequality (25) to read:

$$U_s(v_s) \geq \begin{cases} U_s(b_s) + \int_{v_s}^{b_s} \overline{\pi}_s(t_s)dt_s + \gamma(\overline{v}_s - \underline{v}_s) & \text{for } v_s \leq \underline{v}_s \\ U_s(b_s) + \int_{v_s}^{b_s} \overline{\pi}_s(t_s)dt_s & \text{for } v_s > \underline{v}_s \end{cases} \tag{37}$$

Furthermore, by choosing the following payment function, we can make sure that (37) is satisfied with equality

$$x(v_s, v_b) = \begin{cases} \chi_b(v_b) - \chi_s(v_s) + K + \gamma(\overline{v}_s - \underline{v}_s) & \text{for } v_s \leq \underline{v}_s \\ \chi_b(v_b) - \chi_s(v_s) + K & \text{for } v_s > \underline{v}_s \end{cases}, \tag{38}$$

This statement follows straightforwardly from the same calculations as in the proof of the *if* part of theorem 3. To see that the payment function remains IC, note that for the buyer and $v_s, v_s' \leq \underline{v}_s$ or $v_s, v_s' > \underline{v}_s$ the addition of a constant on to payment function makes no difference for incentive compatibility. For $v_s' \leq \underline{v}_s < \overline{v}_s \leq v_s$, we have

$$\begin{aligned} &U_s(v_s, v_s) - U_s(v_s', v_s) \\ &\qquad = - v_s(\overline{p}(v_s) - \overline{p}_s(v_s')) - \chi_s(v_s) + \chi_s(v_s') - \gamma(\overline{v}_s - \underline{v}_s) \\ &\qquad \geq \int_{t_s = v_s'}^{v_s} (v_s - t_s) d[-\overline{\pi}_s(t_s)] - (\overline{p}_s(\overline{v}_s) - \overline{p}_s(\underline{v}_s))(\overline{v}_s - \underline{v}_s) \geq 0 \, , \end{aligned} \tag{39}$$

where the first inequality follows from the upper limit we imposed on $\gamma$, and the last one from the fact that $\overline{p}_s(\cdot)$ is a non-increasing function. It is easy to verify this payment function results in (37) being satisfied with equality.

Retracing the steps that lead up to (27) in the proof of Theorem 3, we can then arrive at a modified condition:

$$G(\gamma) \equiv \int_{a_b}^{b_b} v_b \overline{p}(v_b) dF_b - \int_{a_s}^{b_s} v_s \overline{p}(v_s) dF_s - \gamma(\overline{v}_s - \underline{v}_s) F_s(\underline{v}_s)$$
$$- \int_{a_s}^{b_s} F_s(t_s) \overline{\pi}_s(t_s) dt_s - \int_{a_b}^{b_b} (1 - F_b(t_b)) \overline{\pi}_b(t_b) dt_b = U_s(b_s) + U_b(a_b) \geq 0 \,, \tag{40}$$

where we have defined the left hand side as $G(\gamma)$.

Now, the second-best mechanism is given by maximizing the aggregate welfare subject to (40), i.e., maximizing the Lagrangian through the choice of $p(\cdot, \cdot)$ and $\gamma$:

$$L = \int_{a_b}^{b_b} \int_{a_s}^{b_s} (v_b - v_s) p(v_s, v_b) dF_s dF_b + \lambda G(\gamma) \,, \tag{41}$$

where $\lambda \geq 0$ is a Lagrange multiplier. But since $G(\gamma)$ is decreasing in $\gamma$ and $\gamma$ is bounded by zero from below (by the envelope theorem), the maximum of the Lagrangian requires $\gamma$ to be zero, which finishes the proof. □

*Proof.* **Proof of Lemma 4** To prove Lemma 4, we will focus on a gap of the seller's candidate mixed investment equilibrium; similar arguments apply for the buyer. First, note that the mixing condition for the seller is given by:

$$v_s + U_s(v_s) - c_s(v_s) = v_s' + U_s(v_s') - c_s(v_s') \,, \tag{42}$$

for any $v_s$, $v_s'$ in the support of the seller's mixed strategy. Hence, for a pair of values $\underline{v}_s$, $\overline{v}_s$ bordering a gap, we must have $U_s(\underline{v}_s) - U_s(\overline{v}_s) = \overline{v}_s - \underline{v}_s + c_s(\underline{v}_s) - c_s(\overline{v}_s)$. Dividing by $\overline{v}_s - \underline{v}_s$, we get:

$$1 + \frac{c_s(\underline{v}_s) - c_s(\overline{v}_s)}{\overline{v}_s - \underline{v}_s} = \frac{U_s(\underline{v}_s) - U_s(\overline{v}_s)}{\overline{v}_s - \underline{v}_s} = \overline{p}_s(\overline{v}_s) , \tag{43}$$

where the last equality follows from Lemma 3 for a second-best mechanism. From the convexity of the cost function, we have:

$$1 + \frac{c_s(\underline{v}_s) - c_s(\overline{v}_s)}{\overline{v}_s - \underline{v}_s} = \overline{p}_s(\overline{v}_s) < 1 + c_s'(\overline{v}_s) , \tag{44}$$

Now, consider a deviation from a candidate mixed-strategy equilibrium where a seller invests at $\overline{v}_s - \epsilon$, but reports $\overline{v}_s$. For small $\epsilon$, the expected change in payoff from this deviation is given by:

$$(1 - \overline{p}_s(\overline{v}_s) + c_s'(\overline{v}_s))\epsilon > 0 , \tag{45}$$

meaning that such a deviation will be profitable. Hence, the candidate equilibrium is not an equilibrium strategy. This implies there cannot be any gaps in the seller's equilibrium investment strategy. □